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ON THERMAL STRESSES IN BEAMS: SOME LIMITATIONS OF THE ELEMENTARY THEORY†

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Abstract—Elementary calculations for the axial thermal stresses in beams are compared with those of an exact theory, and estimates of the difference between the two are given for several important special cases, including that of thin-walled beams. It is found that the error in the elementary theory is quite large in certain cases (for example that of a circular section with axisymmetric temperature).

1. INTRODUCTION

THE calculation of normal axial thermoelastic stresses in a free beam is usually carried out on the basis of the elementary formula [1, p. 310]:

$$\sigma_x = -\alpha ET + \frac{P_T}{A} + \frac{M_T y}{I} \tag{1}$$

where T is the temperature,

$$P_T = \int_A \alpha ET \, \mathrm{d}A; \qquad M_T = \int_A \alpha ETy \, \mathrm{d}A \tag{1a}$$

and the other symbols have the usual meanings. Equation (1) holds for arbitrary cross sectional shapes and temperature variations, provided that the y and z axes are centroidal and that the z axis is chosen so as to coincide with the "neutral axis", i.e. the line (which always exists) for which $\int_A \alpha ETz \, \mathrm{d}A = 0$. If these axes are differently selected, a somewhat more complicated formula results, but their adoption implies considerable simplification and no loss of generality. It is furthermore known that corrections to equation (1) must be added in cases in which the temperature is not linear in x, and that these may at times be of some importance, but these will not be the subject of the present study.

The present work will be rather concerned with the fact that, for beams of arbitrary cross-section, the axial stress should be calculated, not from equation (1), but from [1, p. 329]:

$$\sigma_x^* = -\alpha ET^* + \frac{P_T^*}{A} + \frac{M_T^* y}{I} \tag{2}$$

where

$$T^* = T - \frac{\nu}{\alpha E}\nabla^2\phi; \qquad P_T^* = \int_A \alpha ET^* \, \mathrm{d}A; \qquad M_T^* = \int_A \alpha ET^* y \, \mathrm{d}A \tag{2a}$$

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and where $\phi(y, z)$ is an Airy stress function satisfying the plane-strain field equation

$$\nabla^4\phi = -\frac{\alpha E}{1-\nu}\nabla^2 T, \qquad \nabla^2 = \frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{3}$$

and the boundary conditions for traction-free surfaces, namely

$$\phi = \frac{\partial \phi}{\partial n} = 0. \tag{3a}$$

Little information is available concerning the importance of performing the calculations on the basis of T^* rather than of T, other than some results pertaining to thin-walled sections (which will be referred to below) and the obvious fact that $T^* \equiv T$, and thus $\sigma_x \equiv \sigma_x^*$, if T is plane harmonic. It is the purpose of the present note to examine this question in some detail, indicating some useful estimates and simplifications.

In general we note that if upper and lower bounds are determined in any one problem in such a manner that

$$B_L \leq \frac{\nu\nabla^2\phi}{\alpha E} \equiv T - T^* \leq B_U \tag{4}$$

then the error in the elementary theory, i.e. difference between σ_x and σ_x^*, cannot exceed the value given by [2]

$$|\sigma_x^* - \sigma_x| \leq k\alpha E(B_U - B_L) \tag{4a}$$

where k is a numerical factor dependent on the beam cross-section. Values of k for several different shapes are given in [2, 3]; it may be noted here for future reference that for a rectangular section $k = 4/3$. Similarly, if it is known that

$$\frac{\nu|\nabla^2\phi|}{\alpha E} \equiv |T^* - T| \leq B, \tag{5}$$

where B is equal to the larger one of $|B_U|$ and $|B_L|$, then

$$|\sigma_x^* - \sigma_x| \leq k\alpha EB. \tag{5a}$$

The bound of equation (4a) will be better than that of equation (5a) when B_U and B_L are of the same sign, while the opposite is true when $B_U > 0$ and $B_L < 0$. Bounds to be used in conjunction with either equations (4) or (5) will be discussed in what follows. It will first be shown in the next section that an alternative manner of obtaining bounds on the error in terms of B_U, B_L and B, without the introduction of the coefficients k, is always available.

2. CALCULATION OF P_T^* AND M_T^*

It will now be shown that in all cases

$$P_T^* = P_T \qquad \text{and} \qquad M_T^* = M_T \tag{6}$$

and thus

$$\sigma_x^* = \sigma_x + \nu\nabla^2\phi. \tag{6a}$$

It follows that, whenever bounds of the type of equation (4) are known, then

$$\alpha E B_L \leq \sigma_x^* - \sigma_x \leq \alpha E B_U \tag{7}$$

and that, whenever bounds of the type of equation (5) are available, then

$$|\sigma^* - \sigma_x| \leq \alpha E B. \tag{8}$$

Since $k \geq 1$ always, equation (8) gives a better bound than equation (5a), while the relative merits of equations (7) and (4a) depend on the relative magnitudes of B_U, B_L and $B_U - B_L$.

To prove the validity of equations (6), we note that in any problems pertaining (like the ones presently considered) to traction-free bodies, the average value of the first stress-tensor invariant $\sigma_x + \sigma_y + \sigma_z$ is zero [1, p. 306]. Inspection of the proof shows that this statement holds for two-dimensional problems as well, and thus

$$P^* - P_T = \nu\alpha E \int_A \nabla^2\phi \, \mathrm{d}A = \nu A E \int_A (\sigma_y + \sigma_z) \, \mathrm{d}A = 0. \tag{9}$$

Further examination of the proof indicates however that

$$\int_V (\sigma_x + \sigma_y + \sigma_z)\varepsilon \, \mathrm{d}V = 0 \tag{10}$$

for any strain field derivable from a continuously differentiable displacement field in such a manner that

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{i,j}) = \begin{cases} \varepsilon & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{10a}$$

The preceding result follows from $u_i = x_i$ and $\varepsilon = 1$, while if we set

$$u_i = x_k x_i - \delta_{ki} x_j x_j/2 \quad \text{or} \quad \varepsilon = x_k; \tag{11}$$

with any fixed k, we obtain in general

$$\int_V (\sigma_x + \sigma_y + \sigma_z) x_k \, \mathrm{d}V = 0. \tag{11a}$$

Since this result again holds also in two-dimensional problem, the second of equations (6) follows immediately. Furthermore, it is also obvious that the "neutral axis" is the same for either σ_x or σ_x^*.

The same results could alternatively have been derived by noting that the tractions acting within the beam on either side of a plane parallel to the xy or the xz plane must be self-equilibrating.

3. CIRCULAR BAR UNDER $T = T(r)$

The general solution for the plane-strain problem of a bar of circular cross-section, solid or hollow, under an axisymmetric temperature distribution is well known [1, p. 290]. It is easily verified from that solution that

$$\nabla^2\phi = \frac{\alpha E}{1-\nu}\left(-T + \frac{P_T}{A}\right) \tag{12}$$

and that therefore

$$\sigma_x^* = \frac{\alpha E}{1-\nu}\left(-T+\frac{P_T}{A}\right) = \frac{\sigma_x}{1-\nu}. \tag{12a}$$

In this case therefore neglect of the term $(\nu\nabla^2\phi)$ in equation (2) is clearly not permissible unless Poisson's ratio is extremely small. The error is

$$\frac{\sigma_x^*-\sigma_x}{\sigma_x} = \frac{\nu\nabla^2\phi}{\sigma_x} = \frac{\nu}{1-\nu}. \tag{13}$$

It will be shown below that often this is the maximum error to be expected, when referred to the maximum stress $\sigma_{x\max}$.

4. RECTANGULAR BEAM WITH $\nabla^2 T$ = CONSTANT

In this case it is convenient to employ the analogy between the boundary-value problem for ϕ, equations (3), and the one for the deflection w of a clamped plate under the equivalent load p^* given by

$$\frac{P^*}{D} = -\frac{\alpha E\nabla^2 T}{1-\nu} \tag{14}$$

since the solution for constant p^* is well-known [4, 5]. Write the moments M_x and M_y in the plate in the form

$$M_x(x,y) = m_x(x,y)p^*a^2; \qquad M_y(x,y) = m_y(x,y)p^*a^2 \tag{14a}$$

where a is the shorter side of the rectangle $|x| < a/2, |y| < b/2$ and where m_x and m_y are dimensionless coefficients whose values are tabulated for example in the two last-cited references;† then

$$\nabla^2\phi = -\frac{\alpha E\nabla^2 T}{(1-\nu^2)}(m_x+m_y)a^2; \qquad \nabla^2 T = \text{const.} \tag{15}$$

To obtain a quantitative estimate of the magnitude of the correction term in a typical example, let

$$T = T_0(y/b)^2 + T_1(y/b) + T_2 \tag{16}$$

where $T_{0,1,2}$ are constants so that

$$\sigma_x = \alpha E T_0\left[\left(\frac{y}{b}\right)^2 - \frac{1}{12}\right]; \qquad \nabla^2 T = \frac{2T_0}{b^2}. \tag{16a}$$

Numerical values of σ_x and σ_x^* are plotted in Fig. 1 for $\nu = 0{\cdot}3$; they show that:

(a) The maximum value of $\nu\nabla^2\phi$ occurs on $y = 0$, so that considerable error is noted on the maximum tensile stress (i.e. the stress with the same sign as T_0).

(b) The error is smaller at $|y| = b/2$, i.e. on the maximum compressive stress (i.e. the stress with the opposite sign to T_0).

† The values given are easily extended by noting, for example, that $m_y = \nu m_x$ along $|x| = a/2$ and $m_x = \nu m_y$ along $|y| = b/2$, because of equations (3a).

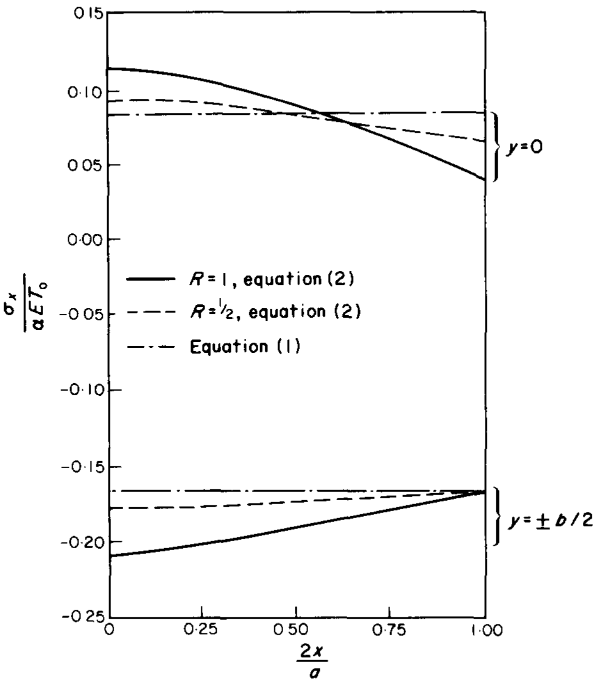

FIG. 1. Stresses in rectangular beams.

(c) In all cases

$$\frac{|\sigma_x^* - \sigma_x|_{max}}{|\sigma_x|_{max}} \leq \frac{\nu}{1-\nu} \tag{16b}$$

as that the error, in this form, never exceeds that of equation (13). Indeed the left-hand side of (16b) is largest for a square-cross-section, in which case, for $\nu = 0{\cdot}3$ it is 0·264, while $\nu/(1-\nu) = 0{\cdot}429$.

(d) The error decreases as $a/b = R$ decreases. For small R (say $R < 1/2$), it can in general be estimated, by use of Grashof's approximate formula [5], which gives:

$$\nabla^2\phi = \frac{\alpha E \nabla^2 T}{(1-\nu)} \frac{(a^2-12x^2)(b^2-4y^2)^2+(b^2-12y^2)(a^2-4x^2)^2}{24(a^4+b^4)} \tag{17}$$

or

$$|\nabla^2\phi| \leq \frac{\alpha E|\nabla^2 T|a^2(1+R^2)}{12(1-\nu)(1+R^4)}. \tag{17a}$$

Thus for the temperature of equation (16), we have approximately

$$\frac{|\sigma_x^* - \sigma_x|_{max}}{|\sigma_x|_{max}} \leq \frac{\nu R^2(1+R^2)}{(1-\nu)(1+R^4)} \tag{17b}$$

so that indeed the error approaches zero for very thin sections. This case can however be treated more generally, i.e. for arbitrary temperatures, as is done next.

5. THIN-WALLED SECTIONS

For thin-walled beams of arbitrary cross-section of (not necessarily constant) thickness $h \ll S$, where S is the developed length of the median line, we can solve for ϕ approximately to obtain [1, p. 334]:

$$\nabla^2\phi = \frac{\alpha E}{1-\nu}\left[-T+\frac{1}{h}\int_{-h/2}^{h/2} T\,\mathrm{d}n+\frac{12n}{h^3}\int_{-h/2}^{h/2} Tn\,\mathrm{d}n\right] \tag{18}$$

correctly to first-order terms in the ratio h/S. Here n is the distance measured normal to the median line $n = 0$ of the cross-section. Since the right-hand side is of the same form as expression (1) for σ_x in a rectangular section, we know immediately that the correction required for thin-walled beams in zero if T is independent of or linear with n. In other cases its magnitude will depend on the maximum temperature variation ΔT across the thickness, and in fact [2]

$$\frac{(1-\nu)|\nabla^2\phi|}{\alpha E|\Delta T|} \le \begin{Bmatrix} \frac{1}{4}\left(2+6\frac{|n|}{h}+\frac{h}{3|n|}\right) & \text{for } \frac{1}{3} \le \frac{2|n|}{h} \le 1 \\ 1 & \text{for } \frac{2|n|}{h} \le \frac{1}{3} \end{Bmatrix} \le \frac{4}{3}. \tag{19}$$

Other bounds can be written, as noted in [2], for other special cases, such as those of temperatures symmetrical or antisymmetrical about $n = 0$. Clearly all these bounds are of the type of those of equation (5) and therefore $|\sigma_x^* - \sigma_x|$ is immediately given by equation (8), with $B = (4/3)\Delta T[\nu/(1-\nu)]$.

In the particular example of the temperature of equation (16), but with T_0 a function of n such that

$$T_M - \Delta T \le T_0(n) \le T_M \tag{20}$$

then, with the bound of equation (19) and with $T_0 = T_M/4$,

$$\frac{|\sigma_x^* - \sigma_x|}{|\sigma_x|} \le \frac{32\nu\Delta T}{(1-\nu)T_M}. \tag{20a}$$

6. ELLIPTIC BEAM, $\nabla^2 T = $ CONSTANT

As for rectangular beams in Section 4, we obtain from the known clamped plate solution [4]

$$\nabla^2\phi = -\frac{\alpha E\nabla^2 T}{1-\nu}a^2\left[\frac{(3+R^2)\xi^2+(3R^2+1)\eta^2-(1+R^2)}{6(1+R^4)+4R^2}\right] \tag{21}$$

for the ellipse

$$\xi^2+\eta^2 = 1 \tag{21a}$$

where

$$\xi = x/a; \qquad \eta = y/b; \qquad R = a/b \le 1. \tag{21b}$$

The largest value of $\nabla^2\phi$ occurs at the end of the minor axis, i.e. at (1,0), or

$$|\nabla^2\phi| \leq \frac{\alpha E|\nabla^2 T|}{1-\nu}\frac{a^2}{3(1+R^4)+2R^2}. \tag{21c}$$

For any particular temperature distribution the error may be computed as was previously done and in general, equations (5) and (8) are once again applicable.

For the temperature of equation (16), we have

$$\sigma_x = \alpha E T_0\left(\frac{1}{4}-\frac{y^2}{b^2}\right); \qquad \frac{|\sigma_x^*-\sigma_x|_{\max}}{|\sigma_x|_{\max}} = \frac{8\nu R^2}{3(1-\nu)[3(1+R^4)+2R^2]} \tag{22}$$

while if the temperature is

$$T = T_0\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) \tag{23}$$

we have

$$\sigma_x = \alpha E T_0\left(\frac{1}{2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right); \qquad \frac{|\sigma_x^*-\sigma_x|_{\max}}{|\sigma_x|_{\max}} = \frac{4\nu(1+R^2)}{(1-\nu)[3(1+R^4)+2R^2]}. \tag{23a}$$

It may also be verified that if the above results are expanded in powers of R, the terms predicted by the thin-walled theory of the preceding article are identical with the first term of the series, i.e. those valid for very small R.

7. ARBITRARY SECTIONS, $\nabla^2 T$ ONE-SIGNED

In the important class of problems in which a beam is monotonically heated on its surface it is easily shown (cf. for example [6]) that the maximum temperature T_M occurs on the surface and that

$$\kappa\nabla^2 T = \frac{\partial T}{\partial t} \geq 0 \tag{24}$$

in this case we can show that the maximum value of $\nabla^2\phi$ also occurs on the surface and cannot exceed that given by

$$\nabla^2\phi \leq \frac{\alpha E T_M}{1-\nu} \tag{25}$$

although no lower bound is readily available. Thus the error on the maximum tensile stress can be predicted, though that on the maximum compressive stress cannot. The latter could be considerably larger than the former if stress concentrations due to notches or holes were present; it was seen not to be so in the preceding examples of the rectangle and the ellipse. The preceding remarks hold if $T > 0$, while if $T < 0$ they must be modified by reversing the sign of the stress.

To prove the validity of (25) we reason as follows. The quantity $[\nabla^2\phi+\alpha ET/(1-\nu)]$ is harmonic and has therefore its maximum value on the surface. As we have seen, however, so does T; hence, if it can be shown that $\nabla^2\phi$ is opposite in sign to T there, then (for $T > 0$, to be specific)

$$\nabla^2\phi + \frac{\alpha ET}{1-\nu} \leq \frac{\alpha E T_M}{1-\nu} \tag{26}$$

and (25) follows directly. To prove that $\nabla^2\phi$ and T are opposite in sign, it is convenient to refer again to the analogous clamped plate, both because of its perhaps easier visualization and because some auxiliary conditions (such as those of equilibrium) are more readily introduced. It is then required to show that for a clamped plate of arbitrary (simply connected) plan form, under an arbitrarily distributed non-negative (say) transverse load $p^* \leq P_M$, the quantity $\nabla^2 w$ cannot be negative at the edge. If w were positive at all interior points of the plate, then clearly at the edge we would need $\partial^2 w/\partial n^2 \geq 0$, with n the interior normal; since $w = 0$ at the edge, this would imply that $\nabla^2 w \geq 0$ also. Hence if $\nabla^2 w$ is to be negative, w must somewhere be negative near the edge, but cannot be so everywhere in the plate because the applied load must do positive work. There would then have to be a $w = 0$ contour (cf. Fig. 2) which, together with a portion of the edge, encloses a region of $w < 0$.

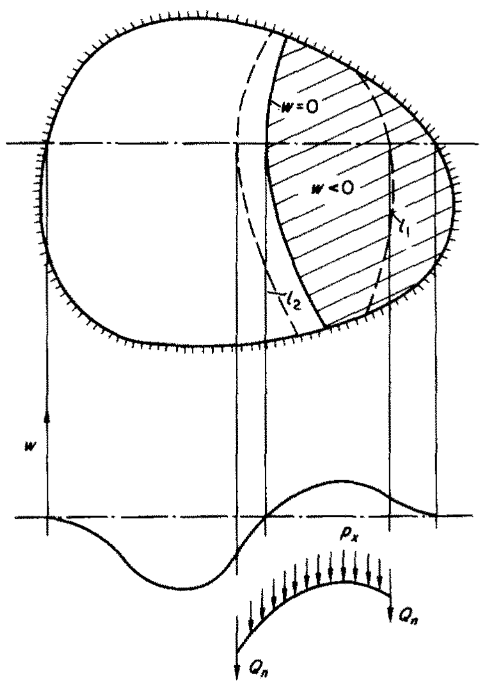

FIG. 2. Deflections of a clamped plate.

Within this region however must be at least one locus of points of inflection (l_1 in Fig. 2), i.e. where $\partial^2 w/\partial n^2 = 0$, and along which $\partial^3 w/\partial n^3 \geq 0$. This implies that $\partial M_n/\partial n = Q_n \geq 0$ there, where Q is the effective transverse shear. Similarly, another locus of inflection points (l_2 in Fig. 2) must exist where $\partial M_n/\partial n = Q_n \leq 0$, with n always indicating the interior normal. The region enclosed by l_1 and l_2 (which is necessarily closed since every line which cuts the $w = 0$ contour has two inflection points at least) cannot be in transverse force equilibrium, as indicated in Fig. 2. It follows that w must be everywhere positive in the interior, and the desired result is proved.

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Абстракт—Сравниваются элементарные расчеты с точными, для осевых термических напряжений в балках. Даются оценки разниц между двумя расчетами, для некоторых важных специальных случаев, учитывающих также тонкостенные балки. Оказывается, что погрешность в элементарной теории очень большая для некоторых случаев, на пример для круглого сечения с осесимметрической температурой.